

# A LINEARIZED KURAMOTO-SIVASHINSKY PDE VIA AN IMAGINARY-BROWNIAN-TIME-BROWNIAN-ANGLE PROCESS

HASSAN ALLOUBA

**ABSTRACT.** We introduce a new imaginary-Brownian-time-Brownian-angle process, which we also call the linear-Kuramoto-Sivashinsky process (LKSP). Building on our techniques in two recent articles involving the connection of Brownian-time processes to fourth order PDEs, we give an explicit solution to a linearized Kuramoto-Sivashinsky PDE in  $d$ -dimensional space:  $u_t = -\frac{1}{8}\Delta^2 u - \frac{1}{2}\Delta u - \frac{1}{2}u$ . The solution is given in terms of a functional of our LKSP.

## 1. STATEMENTS AND DISCUSSIONS OF RESULTS.

One of the prominent equations in modern applied mathematics is the celebrated Kuramoto-Sivashinsky (KS) PDE. This nonlinear equation has generated a lot of interest in the PDE literature (see e.g., [9, 10, 11, 12, 21] and many other papers). In the field of stochastic processes, a great deal of interest is directed at the study of processes in which time is replaced in one way or another by a Brownian motion, and this interest has picked up considerably (see e.g., [1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]) after the fundamental work of Burdzy on iterated Brownian motion ([7, 8]). In [1, 2], we provided a unified framework for such iterated processes (including the IBM of Burdzy) and introduced several interesting new ones, through a large class of processes that we called Brownian-time processes. We then related them to different fourth order PDEs. In this article, and as announced in [2], we modify our process in Theorem 1.2 [2] and build on our methods in [2] to give an explicit solution to a linear version of the KS PDE. One modification needed is the introduction of  $i = \sqrt{-1}$  in both the Brownian-time and the Brownian-exponential, and that leads to a new process we call imaginary-Brownian-time-Brownian-angle process IBTBAP, starting at  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ :

$$(1.1) \quad \mathbb{A}_B^{f,X}(t, x) \triangleq \begin{cases} f(X^x(iB(t))) \exp(iB(t)), & B(t) \geq 0; \\ f(iX^{-ix}(-iB(t))) \exp(iB(t)), & B(t) < 0; \end{cases}$$

where  $X^x$  is an  $\mathbb{R}^d$ -valued Brownian motion starting from  $x \in \mathbb{R}^d$ ,  $X^{-ix}$  is an independent  $i\mathbb{R}^d$ -valued BM starting at  $-ix$  (so that  $iX^{-ix}$  starts at  $x$ ), and both are independent of the inner standard  $\mathbb{R}$ -valued Brownian motion  $B$  starting from 0. The time of the outer Brownian motions  $X^x$  and  $X^{-ix}$  is replaced by an imaginary

---

*Date:* 5/20/2002.

1991 *Mathematics Subject Classification.* Primary 35C15, 35G31, 35G46, 60H30, 60G60, 60J45, 60J35; Secondary 60J60, 60J65.

*Key words and phrases.* Brownian-time Brownian sheet, nonlinear fourth order coupled PDEs, linear systems of fourth order coupled PDEs, Brownian-time processes, initially perturbed fourth order PDEs, Brownian-time Feynman-Kac formula, iterated Brownian sheet, iterated Brownian sheet, random fields.

positive Brownian time; and, when  $f$  is real-valued as we will assume here, the angle of  $\mathbb{A}_B^{f,X}(t, x)$  is the Brownian motion  $B$ . We think of the imaginary-time processes  $\{X^x(is), s \geq 0\}$  and  $\{iX^{-ix}(-is), s \leq 0\}$  as having the same complex Gaussian distribution on  $\mathbb{R}^d$  with the corresponding complex distributional density

$$p_{is}^{(d)}(x, y) = \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is}.$$

We will also call the process given by (1.1) the  $d$ -dimensional Linear-Kuramoto-Sivashinsky process (LKSP) starting at  $f$  (clearly  $\mathbb{A}_B^{f,X}(0, x) = f(x)$ ). The dimension in  $d$ -dimensional IBTBAP (or  $d$ -dimensional LKSP) refers to the dimension of the BMs  $X^x$  and  $X^{-ix}$ , which is also the dimension of the spatial variable in the associated linearized KS PDE as we will see shortly.

Now, motivated by the definitions of  $v_\epsilon$  and  $u_\epsilon$  in the proof of Theorem 1.2 in [2], we let

$$(1.2) \quad \begin{aligned} v(s, x) &\triangleq \exp(is) \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is} dy \\ u(t, x) &\triangleq \int_{-\infty}^0 v(s, x) p_t(0, s) ds + \int_0^\infty v(s, x) p_t(0, s) ds \end{aligned}$$

where  $p_t(0, s)$  is the transition density of the inner (one-dimensional) Brownian motion  $B$ :

$$p_t(0, s) = \frac{1}{\sqrt{2\pi t}} e^{-s^2/2t}.$$

We may think of  $v$  and  $u$  in terms of complex expectation by defining  $v(s, x) \triangleq \mathbb{E}^{\mathbb{C}}[f(X^x(is)) \exp(is)]$  and  $u(t, x) \triangleq \mathbb{E}^{\mathbb{C}}[\mathbb{A}_B^{f,X}(t, x)]$ . A more detailed study of the rich connection between our process and its complex distribution to the KS PDE and its implications is the subject of an upcoming article [3]. We are now ready to state our main result.

**Theorem 1.1.** *Let  $f \in C_c^2(\mathbb{R}^d; \mathbb{R})$  with  $D_{ij}f$  Hölder continuous with exponent  $0 < \alpha \leq 1$ , for all  $1 \leq i, j \leq d$ . If  $u(t, x)$  is given by (1.2) then  $u(t, x)$  solves the linearized Kuramoto-Sivashinsky PDE*

$$(1.3) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = -\frac{1}{8} \Delta^2 u(t, x) - \frac{1}{2} \Delta u(t, x) - \frac{1}{2} u(t, x), & t > 0, x \in \mathbb{R}^d; \\ u(0, x) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

## 2. PROOF OF THE MAIN RESULT

**Proof of Theorem 1.1.** Let  $u$  and  $v$  be as given in (1.2). Differentiating  $u(t, x)$  with respect to  $t$  and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that  $p_t(0, s)$  satisfies the heat equation

$$\frac{\partial}{\partial t} p_t(0, s) = \frac{1}{2} \frac{\partial^2}{\partial s^2} p_t(0, s)$$

and integrating by parts twice using the fact that the boundary terms vanish at  $\pm\infty$  and that  $(\partial/\partial s)p_t(0, s) = 0$  at  $s = 0$ , we obtain

$$\begin{aligned}
(2.1) \quad \frac{\partial}{\partial t}u(t, x) &= \int_{-\infty}^0 v(s, x) \frac{\partial}{\partial t}p_t(0, s)ds + \int_0^\infty v(s, x) \frac{\partial}{\partial t}p_t(0, s)ds \\
&= \frac{1}{2} \left[ \int_{-\infty}^0 v(s, x) \frac{\partial^2}{\partial s^2}p_t(0, s)ds + \int_0^\infty v(s, x) \frac{\partial^2}{\partial s^2}p_t(0, s)ds \right] \\
&= \frac{1}{2}p_t(0, 0) \left[ \left( \frac{\partial}{\partial s}v(s, x) \right) \Big|_{s=0^-} + \left( \frac{\partial}{\partial s}v(s, x) \right) \Big|_{s=0^+} \right] \\
&\quad + \frac{1}{2} \int_{-\infty}^0 p_t(0, s) \frac{\partial^2}{\partial s^2}v(s, x)ds + \frac{1}{2} \int_0^\infty p_t(0, s) \frac{\partial^2}{\partial s^2}v(s, x)ds \\
&= \frac{1}{2} \int_{-\infty}^0 p_t(0, s) \left[ -\frac{1}{4}\Delta^2 v(s, x) - \Delta v(s, x) - v(s, x) \right] ds \\
&\quad + \frac{1}{2} \int_0^\infty p_t(0, s) \left[ -\frac{1}{4}\Delta^2 v(s, x) - \Delta v(s, x) - v(s, x) \right] ds \\
&= -\frac{1}{8}\Delta^2 u(t, x) - \frac{1}{2}\Delta u(t, x) - \frac{1}{2}u(t, x)
\end{aligned}$$

where for the last two equalities in (2.1) we have used the fact that

$$\begin{aligned}
(2.2) \quad \frac{\partial v}{\partial s} &= \frac{i}{2}\Delta v(s, x) + iv(s, x) \\
\frac{\partial^2 v}{\partial s^2} &= -\frac{1}{4}\Delta^2 v(s, x) - \Delta v(s, x) - v(s, x),
\end{aligned}$$

and the conditions on  $f$  to take the applications of the derivatives outside the integrals in (2.1) and (2.2) (the steps of Lemma 2.1 in [2] easily translates to our setting here, see the discussion below). Clearly  $u(0, x) = f(x)$ , and the proof is complete.  $\square$

As we indicated above, only minor changes to Lemma 2.1 in [2] are needed to justify pulling the derivatives outside the integrals in (2.1) under the conditions on  $f$  of Theorem 1.1. We now adapt Lemma 2.1 [2] to our setting here, and we point out the necessary changes in its proof:

**Lemma 2.1.** *Let  $v(s, x)$  be given by (1.2) and let  $f$  be as in Theorem 1.1. Let*

$$(2.3) \quad u_1(t, x) \triangleq \int_{-\infty}^0 v(s, x)p_t(0, s)ds \quad \text{and} \quad u_2(t, x) \triangleq \int_0^\infty v(s, x)p_t(0, s)ds,$$

*then  $\Delta^2 u_1(t, x)$  and  $\Delta^2 u_2(t, x)$  are finite and*

$$(2.4) \quad \Delta^2 u_1(t, x) = \int_{-\infty}^0 \Delta^2 v(s, x)p_t(0, s)ds \quad \text{and} \quad \Delta^2 u_2(t, x) = \int_0^\infty \Delta^2 v(s, x)p_t(0, s)ds.$$

*Proof.* As in the proof of Lemma 2.1 [2], letting  $\mathring{\mathbb{R}}_+ = (0, \infty)$  and  $\mathring{\mathbb{R}}_- = (-\infty, 0)$ , it suffices to show

$$(2.5) \quad \frac{\partial^4}{\partial x_j^4} \int_{\mathring{\mathbb{R}}_\pm} v(s, x)p_t(0, s)ds = \int_{\mathring{\mathbb{R}}_\pm} \frac{\partial^4}{\partial x_j^4} v(s, x)p_t(0, s)ds, \quad j = 1, \dots, d.$$

Letting  $p_{is}^{(d)}(x, y) = (2\pi is)^{-d/2} e^{-|x-y|^2/2is}$  and using the conditions on  $f$ , we easily get

$$(2.6) \quad \begin{aligned} \frac{\partial^4}{\partial x_j^4} v(s, x) p_t(0, s) &= \exp(is) \left( \int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial y_j^4} p_{is}^{(d)}(x, y) dy \right) p_t(0, s) \\ &= \exp(is) \left( \int_{\mathbb{R}^d} \frac{\partial^2}{\partial y_j^2} f(y) \frac{\partial^2}{\partial y_j^2} p_{is}^{(d)}(x, y) dy \right) p_t(0, s). \end{aligned}$$

Rewriting the last term in (2.6), and letting  $h_j(y) \triangleq \partial^2 f(y) / \partial y_j^2$ , we have

$$(2.7) \quad \begin{aligned} &\left| \exp(is) \left( \int_{\mathbb{R}^d} (2\pi is)^{-d/2} \left( \frac{-(x_j - y_j)^2 + is}{s^2} \right) e^{-|x-y|^2/2is} h_j(y) dy \right) \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \right| \\ &= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \left| \left( \int_{\mathbb{R}^d} (2\pi is)^{-d/2} \left( \frac{-(x_j - y_j)^2 + is}{s^2} \right) e^{-|x-y|^2/2is} (h_j(y) - h_j(x)) dy \right) \right| \\ &\leq \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} (2\pi |s|)^{-d/2} \left| \frac{-(\tilde{x}_j - y_j)^2 + |s|}{s^2} \right| e^{-|\tilde{x}-y|^2/2|s|} |h_j(y) - h_j(\tilde{x})| dy \\ &= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \mathbb{E}_{\mathbb{P}} \left| \left( \frac{(\tilde{x}_j - W_j^{\tilde{x}}(|s|))^2 - |s|}{s^2} \right) (h_j(W^{\tilde{x}}(|s|)) - h_j(\tilde{x})) \right|, \end{aligned}$$

for some  $\tilde{x} \in \mathbb{R}^d$  where  $\tilde{x}_j = \pm x_j$  for  $j = 1, \dots, d$ ; and where  $W^{\tilde{x}} : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$  is a standard Brownian motion starting at  $\tilde{x} \in \mathbb{R}^d$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $W_j^{\tilde{x}}$  is its  $j$ -th component. The inequality in (2.7) follows easily if  $h_j$  is a polynomial, and standard approximation yields the inequality for  $h_j \in C_c(\mathbb{R}^d; \mathbb{R})$ . Now, exactly as in [2] (2.6) and (2.7); we use the Brownian motion scaling for  $W^{\tilde{x}}$ , the Cauchy-Schwarz inequality on the last term in (2.7), and the Hölder condition on  $h_j$  to deduce that the last term in (2.7) is bounded above by  $K \exp(-s^2/2t) / (\sqrt{2\pi t} |s|^{1-\alpha/2}) \in L^1((-\infty, 0), ds) \cap L^1((0, \infty), ds)$ ; hence  $|\partial^4 / \partial x_j^4 v(s, x) p_t(0, s)| \in L^1((-\infty, 0), ds) \cap L^1((0, \infty), ds)$ , which completes the proof by standard analysis.  $\square$

**Acknowledgements.** I'd like to thank Ciprian Foias for his encouragement to pursue this project and for his support. I also enjoyed several one on one fruitful discussions with him. This research is supported in part by NSA grant MDA904-02-1-0083.

## REFERENCES

- [1] Allouba, H., Zheng, W., Brownian-time processes: the PDE Connection and the half-derivative generator. *Ann. of Probab.* 29 (4) (2001) 1780–11795.
- [2] Allouba, H., Brownian-Time Processes: The PDE Connection II and the Corresponding Feynman-Kac Formula. *Trans. Amer. Math. Soc.* 354 (11) (2002) 4627–4637.
- [3] Allouba, H., On the connection between the Kuramoto-Sivashinsky PDE and imaginary-Brownian-time-Brownian-angle processes. In preparation (2003).
- [4] Bass, R., Probabilistic techniques in analysis. Springer-Verlag, New York, 1995.
- [5] Bass, R., Diffusions and elliptic operators. Springer-Verlag, New York, 1997.
- [6] Benachour, S., Roynette, B., Vallois, P., Explicit solutions of some fourth order partial differential equations via iterated Brownian motion. *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996)*, 39–61, *Progr. Probab.*, 45, Birkhäuser, Basel, 1999.
- [7] Burdzy, K., Some path properties of iterated Brownian motion. *Seminar on Stochastic Processes 1992*, Birkhäuser, (1993), 67–87.

- [8] Burdzy, K., Variation of iterated Brownian motion. Workshop and conf. on measure-valued processes, stochastic PDEs and interacting particle systems. CRM Proceedings and Lecture Notes 5, (1994), 35–53.
- [9] Changbing, H., Temam, R., Robust control of the Kuramoto-Sivashinsky equation. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 8 (2001), no. 3, 315–338.
- [10] Cheskidov, A., Foias, C., On the non-homogeneous stationary Kuramoto-Sivashinsky equation. Phys. D 154 (2001), no. 1-2, 1–14.
- [11] Foias, C., Kukavica, I., Determining nodes for the Kuramoto-Sivashinsky equation. J. Dynam. Differential Equations 7 (1995), no. 2, 365–373.
- [12] Foias, C., Nicolaenko, B., Sell, G. R., Temam, R., Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension. J. Math. Pures Appl. (9) 67 (1988), no. 3, 197–226.
- [13] Funaki, T., Probabilistic construction of the solution of some higher order parabolic differential equation. Proc. Japan Acad. Ser. A Math. Sci. 55, no. 5, (1979), 176–179.
- [14] Hochberg, K., Orsingher, E., Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. J. Theoret. Probab. 9, no. 2, (1996), 511–532.
- [15] Jolly, M. S., Rosa, R., Temam, R., Evaluating the dimension of an inertial manifold for the Kuramoto-Sivashinsky equation. Adv. Differential Equations 5 (2000), no. 1-3, 31–66.
- [16] Khoshnevisan, D., Lewis, T., Iterated Brownian motion and its intrinsic skeletal structure. Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996), 201–210, Progr. Probab., 45, Birkhuser, Basel, 1999.
- [17] Khoshnevisan, D., Lewis, T., Stochastic calculus for Brownian motion on a Brownian fracture. Ann. Appl. Probab. 9 (1999), no. 3, 629–667.
- [18] Revuz, D. and Yor, M., Continuous martingales and Brownian motion. Springer, New York, 1999.
- [19] Shi, Z., Lower limits of iterated Wiener processes. Statist. Probab. Lett. 23 (1995), no. 3, 259–270.
- [20] Shi, Z., Yor, M., Integrability and lower limits of the local time of iterated Brownian motion. Studia Sci. Math. Hungar. 33 (1997), no. 1-3, 279–298.
- [21] Temam, R., Wang, X., Estimates on the lowest dimension of inertial manifolds for the Kuramoto-Sivashinsky equation in the general case. Differential Integral Equations 7 (1994), no. 3-4, 1095–1108.

DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT, OHIO 44242

*E-mail address:* allouba@math.kent.edu